motion. One of these is closely given by $1 - \frac{3}{2}\epsilon_3$, where the unit of time is the 1-month lunar orbital period; the other is similarly given by

$$\frac{1}{2\epsilon_3}\left(1 - \frac{\epsilon_1}{\epsilon_3}\right)^{-1/2}$$

Using the numerical values given by (4), these periods correspond very nearly to 1 month and to 870 months, respectively. Whereas the very long-period motion presents obvious obstacles from the point of view of measurements for any significant part of even one oscillation, the shortest-period motion, viz., the coupled nodal-inclination mode of period 1 month, is a distinctly more attractive prospect for observation. The shortness of the period also justifies neglecting effects of long-period forcing functions such as the dominant solar attraction.

In summary, it has been shown that the near-symmetry of lunar mass distribution leads to sharply distinguishable dynamic characteristics, and that the mode of shortest period, almost completely overlooked in the past, is a combined motion in node and inclination, interrelated in an elementary manner. The same distinctions should also prove useful as guides for the construction of Liapunov functions required in the application of direct methods for the nonlinear stability problem, where once again the moon should serve as a shining example in a new class of studies in dynamics.

References

[1] Michelson, I., "Equilibrium orientations of gravity-gradient satellites," AIAA J. 1, 493 (1963).

[2] Danby, J. M. A., *Fundamentals of Celestial Mechanics* (Macmillan Co., New York, 1962), p. 335.

[3] Koziel, K., "Libration of the moon," *Physics and Astronomy of the Moon*, edited by Z. Kopal (Academic Press, New York, 1961).

An Explicit Guidance Concept

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Nomenclature

- A,B = thrust integrals, defined specifically by Eq. (1)
- **F** = thrust vector
- J_1,J_2,J_3 = time integrals defined specifically by Eq. (3)
- m = mass
- **r** = central body radius vector to vehicle
- **S** = slant range vector from target to vehicle
- t = time
- V_j = effective exhaust velocity
- X,Y,Z = inertial, target centered, Cartesian coordinate system
- θ = coangle between thrust vector and Z axis
- μ = central body gravitational constant = gR^2
- ξ = predicted propellant mass fraction to be consumed during burning
- τ = burning time
- ψ = angle between X axis and projection of **F** in X-Y plane
- ω = mean motion = $\{\mu/[\frac{1}{2}(r_0 + r_t)]^3\}^{1/2}$

Superscript

- $(\dot{\ })$ = derivative with respect to time

Subscript

- c = command value
- n = refers to value corresponding to nth step
- 0 = at time zero
- t = at landing site

This development was presented as an Appendix to Ref. 1 at the ARS Lunar Missions Meeting, Cleveland, Ohio, July 17–19, 1962.

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In the text of Ref. 1, a solution was obtained for the differential equation of motion for a particle in a uniform central force field under the influence of a force **F** and of a mass m. This solution is of the form

$$\begin{aligned}\mathbf{r} &= (\mathbf{r}_0 - \mathbf{B})\cos\omega\tau + (1/\omega)(\dot{\mathbf{r}}_0 + \omega\mathbf{A})\sin\omega\tau \\ \dot{\mathbf{r}} &= (\dot{\mathbf{r}}_0 + \omega\mathbf{A})\cos\omega\tau - \omega(\mathbf{r}_0 - \mathbf{B})\sin\omega\tau\end{aligned} \tag{1}$$

where

$$\mathbf{A} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\cos\omega t\, dt$$

$$\mathbf{B} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\sin\omega t\, dt$$

The components of the thrust vector **F** are assumed to have the following time-dependent form:

$$\begin{aligned}F_x(t) &= F(t)\cos(\theta + \dot{\theta}t)\cos(\psi + \dot{\psi}t) \\ F_y(t) &= F(t)\cos(\theta + \dot{\theta}t)\sin(\psi + \dot{\psi}t) \\ F_z(t) &= F(t)\sin(\theta + \dot{\theta}t)\end{aligned} \tag{2}$$

If the thrust is assumed to be constant and $\dot{\theta}$ and $\dot{\psi}$ are assumed to be of the same order as ω, then analytic solutions for **A** and **B** may be achieved. These are given below as Eqs. (3), correct to first order in $\omega\tau$:

$$\left.\begin{aligned}A_x &= (1/\omega)[J_1\cos\theta\cos\psi - \\ &\qquad J_2(\dot{\theta}\sin\theta\cos\psi + \dot{\psi}\sin\psi\cos\theta)] \\ A_y &= 1/\omega[J_1\cos\theta\sin\psi - \\ &\qquad J_2(\dot{\theta}\sin\theta\sin\psi - \dot{\psi}\cos\psi\cos\theta)] \\ A_z &= (1/\omega)(J_1\sin\theta + J_2\dot{\theta}\cos\theta) \\ B_x &= J_2\cos\theta\cos\psi - J_3(\dot{\theta}\sin\theta\cos\psi + \dot{\psi}\sin\psi\cos\theta) \\ B_y &= J_2\cos\theta\sin\psi - J_3(\dot{\theta}\sin\theta\sin\psi - \dot{\psi}\cos\psi\cos\theta) \\ B_z &= J_2\sin\theta + J_3\dot{\theta}\cos\theta \\ J_1 &= -V_j\ln(1 - \xi) \\ J_2 &= (\tau/\xi)(J_1 - V_j\xi) \\ J_3 &= (\tau/\xi)(J_2 - \tfrac{1}{2}V_j\tau\xi)\end{aligned}\right\} \tag{3}$$

where

$$V_j = F/\dot{m} \qquad \xi = \dot{m}\tau/m$$

It is now possible to solve explicitly for $\theta_c, \dot{\theta}_c, \psi_c$, and $\dot{\psi}_c$. The procedure is as follows. Let there be specified some final position and velocity vector, $\mathbf{r}_t$ and $\dot{\mathbf{r}}_t$, which is to be achieved at time = τ. These shall satisfy Eqs. (1). With this substitution, rearrangement yields

$$\begin{aligned}\omega\left(\frac{\mathbf{r}_t}{\cos\omega\tau} - \mathbf{r}\right) &= -\mathbf{B}\omega + (\dot{\mathbf{r}} + \omega\mathbf{A})\tan\omega\tau \\ \left(\frac{\dot{\mathbf{r}}_t}{\cos\omega\tau} - \dot{\mathbf{r}}\right) &= \omega\mathbf{A} + \omega(\mathbf{B} - \mathbf{r})\tan\omega\tau\end{aligned} \tag{4}$$

where, for convenience, the subscript 0 has been dropped.

Equations (4) yield solutions for **A** and **B** as follows:

$$\begin{aligned}\mathbf{A} &= \mathbf{r}_t\sin\omega\tau - 1/\omega\Delta\dot{\mathbf{r}} \\ \mathbf{B} &= \Delta\mathbf{r} + \dot{\mathbf{r}}_t/\omega\sin\omega\tau\end{aligned} \tag{5}$$

where

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_t\cos\omega\tau \qquad \Delta\dot{\mathbf{r}} = \dot{\mathbf{r}} - \dot{\mathbf{r}}_t\cos\omega\tau$$

The values attained from Eqs. (5) for **A** and **B**, of necessity, must have a corresponding value in Eqs. (3). From the third and sixth of Eqs. (3), a solution is obtained for $\sin\theta_c$ and $\dot{\theta}_c$:

$$\sin\theta_c = \frac{\omega A_Z(J_3/J_2) - B_Z}{J_1(J_3/J_2) - J_2} \quad (6)$$

$$\dot{\theta}_c = \frac{B_Z - J_2 \sin\theta_c}{J_3 \cos\theta_c} \quad (7)$$

Similarly, a solution is obtained for $\sin\psi_c$ and $\dot{\psi}_c$ from the second and fifth of Eqs. (3):

$$\sin\psi_c = \frac{1}{\cos\theta_c}\,\frac{\omega A_y(J_3/J_2) - B_y}{J_1(J_3/J_2) - J_2} \quad (8)$$

$$\dot{\psi}_c = \frac{\omega A_y + (\dot{\theta}_c J_2 \sin\theta_c - J_1 \cos\theta_c)\sin\psi_c}{J_2 \cos\theta_c \cos\psi_c} \quad (9)$$

The next logical step in this development is to solve for τ and ξ, using the first and fourth of Eqs. (3) in combination with Eqs. (6–9). Unfortunately, this step cannot be carried through in a straightforward manner. The resulting equations for ξ and τ require the simultaneous solution of two transcendental equations.

Under laboratory conditions, this solution is practical with the aid of a high-speed computer. However, the system being developed here is intended for operational use under anything but laboratory conditions. If the solution did not, for some reason, converge properly under an off-design condition, there would not be time to force convergence nor would there be time to work out new initial values that might result in convergence.

Rather than attempt the solution just described, it is suggested that two approximations for ξ and τ be used. These will result in trajectories that are not flown under constant thrust. However, the approximations are sufficiently accurate to assure that the throttling range required to accommodate the deficiencies in the solution will be small, on the order of 2%. This has been verified with a two-degree of-freedom digital simulation, using Eqs. (6, 7, 10, and 11) for trajectory control.

The approximations for ξ and τ are given as

$$\xi = 1 - \exp\left[-\frac{1}{V_j}\left(|\dot{\mathbf{S}}| + 2g\,\frac{h - h_t}{V + V_t}\right)\right] \qquad (V + V_t) > 0 \quad (10)$$

$$\tau = \left|\frac{S^2}{\dot{\mathbf{S}}\cdot\mathbf{S}\left[\frac{1}{\ln(1-\xi)} + \frac{1}{\xi}\right] + \dot{\mathbf{r}}_t\cdot\mathbf{S}}\right|\frac{1}{K_n} \quad (11)$$

where $\mathbf{S} = \mathbf{r} - \mathbf{r}_t$. The K term in Eq. (11) is found in the following manner: assume that the first time a τ is computed during a flight, $K_{(n-1)} = 1.0$. Compute the thrust level required to satisfy Eqs. (10) and (11) from Eq. (12):

$$F_c = V_j m(\xi/\tau) \quad (12)$$

Now compute $\theta_c, \dot{\theta}_c, \psi_c$, and $\dot{\psi}_c$ from the appropriate equations, and then compute the predicted landing point from Eqs. (1) and (3). A predicted slant range vector is computed as

$$\mathbf{S}_t = \mathbf{r} - \mathbf{r}_t \quad (13)$$

Then K_n is found from Eq. (14):

$$K_n = \left(\frac{\mathbf{S}_t\cdot\mathbf{S}_t}{\mathbf{S}\cdot\mathbf{S}}\right)^{1/2} \quad (14)$$

The K term need not be computed continuously. For most flights, updating every 10 sec is sufficient. The results from the digital simulation and the associated error analysis of this concept are to be reported at a later date.

Reference

[1] Markson, E., Bryant, J., and Bergsten, F., "Simulation of manned lunar landing," *AIAA Progress in Astronautics and Rocketry: Technology of Lunar Exploration*, edited by Clifford I. Cummings and Harold R. Lawrence (Academic Press Inc., New York, 1963), pp. 561–589.

Contour Calculations for Chemical Nonequilibrium Flow

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The problem of a reacting flow expanding through a divergent nozzle has been examined in the light of designing nozzle contours to produce specific interrelationships among the problem variables. Using a Pace Electronic analog computer, solutions have been generated for a variety of parameter restrictions. The results show that hydrodynamically reasonable nozzle shapes may be used to provide a convenient method of experimentally producing arbitrary supersonic nonequilibrium flows.

Nomenclature

a = reciprocal of dimensionless area A
E = dissociation energy
F = dimensionless recombination rate, $4r_0p_0^2\eta k_R/u_0R^2T_0^2T$
G = dimensionless equilibrium constant, $K_p/4\eta p_0 T$
k_R = recombination rate
K = dimensionless term, $4NE_A/RT_0$
K_p = equilibrium constant
M = molecular weight
N = Avogadro's number
p = dimensionless pressure
r = dimensionless streamline coordinate
R = universal gas constant
T = dimensionless temperature
u = dimensionless velocity
α = mass fraction of dissociated atomic species
β = critical throat parameter, $\rho_0 u_0^2/p_0$
Δ = computer time scale factor
η = dimensionless constant, $\beta RT_0/2M_A u_0^2$
ρ = dimensionless mass density
τ = computer time

Subscripts

A = atomic
e = local equilibrium
0 = dimensional throat value

In this note a method for designing hypersonic nozzle contours for specified thermodynamic requirements is presented using the governing equations of pseudo-one-dimensional inviscid flow of an ideal diatomic gas undergoing a dissociation reaction. For nonequilibrium flow, where chemical reactions proceed at finite rates, the flow properties are dependent upon the axial distance traveled. As a result, it is

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